

THE
**MATHEMATICAL
GAZETTE**

EDITED BY
W. J. GREENSTREET, M.A.

WITH THE CO-OPERATION OF
F. S. MACAULAY, M.A., D.Sc.

AND
PROF. E. T. WHITTAKER, M.A., F.R.S.

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C A M B R I D G E U N I V E R S I T Y P R E S S

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ON THE THEORY OF THE PLANE COMPLEX WITH SIMPLE GEOMETRICAL AND KINEMATICAL ILLUSTRATIONS.

By R. W. GENÈSE, M.A.

$OA, OB = x_1, x_2$ being distances, carrying signs, along an axis, the " Bearing " of B with respect to A (called a vector, or step, or displacement) is given by $AB = OB - OA = x_2 - x_1$ in all cases, and is independent of the position of O .

If we wish to add two such steps, a construction is necessary, viz. they are placed " end on " in either order; and the sum is dependent on the position of O ; if M be the mid-point of AB , the sum $= 2OM$.

Now let i denote an operator which will turn a step along x through an angle $+90^\circ$; then i^2 , short for ii , reverses the step; and i is hence commonly, but not necessarily, identified with $+\sqrt{-1}$.

Let x_1, y_1 be the rectangular coordinates of P ; then, extending the idea of " end on " addition, we say that

$$OP = x_1 + iy_1$$

(y_1 being at first supposed to be taken along axis of x , and then rotated), then

$$iOP = ix_1 - y_1.$$

An inspection of the diagram shows that ON, NP have both been turned round O through $+90^\circ$, and therefore also OP .

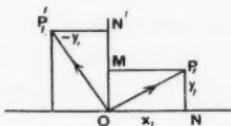
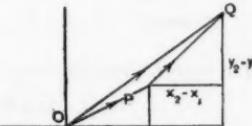


FIG. 1.



Hence the operator i turns any vector through $+90^\circ$.
Let $OQ = x_2 + iy_2$, then the bearing of Q with respect to P is shown by

$$PQ = x_2 - x_1 + i(y_2 - y_1) = OQ - OP.$$

It is geometrically obvious that all vectors equal and parallel to PQ have equal algebraical equivalents, and conversely.

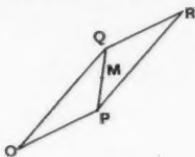


FIG. 2.

Again

$$OP + OQ = x_1 + x_2 + i(y_1 + y_2) \\ = 2(\bar{x} + i\bar{y}) = 2OM.$$

where \bar{x} , \bar{y} are the coordinates of M , the mid-point of PQ .

=OR

the parallelogram $POQR$ having been completed.

Or, thus : if OR is the sum required,

$$OQ = OR - OP \\ \equiv PR.$$

then OQ may be added to OP by placing it "end on" in the position PR .

Ex. 1. Let ABC be any triangle, O its circumcentre, M the mid-point of BC .

$OB + OC = 2OM$, and is perpendicular to BC .

$\therefore OA + OB + OC = OP$, where P is a point on the perpendicular from A on BC and $AP = 2OM$.

But by changing the order of the terms we could show that P is on either of the other perpendiculars of the triangle.

Hence the perpendiculars of a triangle meet in a point whose distances from the vertices are twice the distances of the circumcentre from the sides.

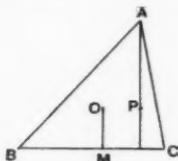


FIG. 3.

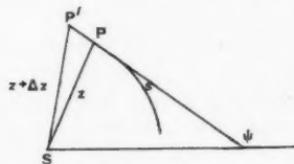


FIG. 4.

If end of vector z describe curve,

$$\Delta z = SP' - SP = PP',$$

Δs =element of arc.

Making $\Delta s \rightarrow 0$,

$$\frac{dz}{ds} = \text{unit vector along tangent at } P \\ = \cos \psi + i \sin \psi = \text{cis } \psi. \quad \dots \dots \dots \quad (1)$$

Ex. 2. In particular, if $z = \text{cis } \theta$, $\psi = \theta + \frac{\pi}{2}$,

$$ds = d\theta \cdot i$$

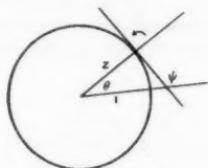


FIG. 5

$$\therefore \frac{dz}{d\theta} = \operatorname{cis}\left(\theta + \frac{\pi}{2}\right) = i \operatorname{cis}\theta. \quad \dots \quad (2)$$

Equating real and imag. we have at once the d.c.'s of $\cos \theta$ and $\sin \theta$.

Ex. 3. Again, if z be a function of time t ,

$$\frac{dz}{dt} = \frac{dz}{ds} \cdot \frac{ds}{dt} = \text{vector } \frac{ds}{dt} \text{ along tangent at } P$$

= velocity, in Thomson and Tait's sense = $v \operatorname{cis} \psi$, v the speed;

$$\therefore \text{total acceleration} = \frac{d}{dt}(v \cos \psi)$$

$$= \frac{dv}{dt} \operatorname{cis} \psi + v \cdot i \operatorname{cis} \psi \frac{d\psi}{ds} \frac{ds}{dt},$$

i.e. $\frac{dv}{dt}$ along tangent and $\frac{v^2}{\rho}$ inwards along normal.

If $z = r \operatorname{cis} \theta$,

$$\frac{dz}{dt} = \frac{dr}{dt} \operatorname{cis} \theta + ri \operatorname{cis} \theta \frac{d\theta}{dt},$$

$$\frac{d^2z}{dt^2} = \frac{d^2r}{dt^2} \operatorname{cis} \theta + 2 \frac{dr}{dt} \cdot i \operatorname{cis} \theta \frac{d\theta}{dt} + r(i)^2 \operatorname{cis} \theta \left(\frac{d\theta}{dt} \right)^2 + ri \operatorname{cis} \theta \frac{d^2\theta}{dt^2},$$

giving the usual polar formulae of dynamics.

Ex. 4. Let P describe a curve under radial retardation varying as z , starting from A at right angles to OA . The equation of motion is

$$\frac{d^2z}{dt^2} = -k^2 z;$$

$$\therefore z = A \cos kt + B \sin kt, \quad \frac{dz}{dt} = -kA \sin kt + kB \cos kt,$$

when $t=0$, $a=A$ and $v_0i=kB$.

∴ coordinates of P are $a \cos kt$ and $\frac{v_0}{k} \sin kt$.

$$\therefore P \text{ describes ellipse} \quad \frac{x^2}{a^2} + \frac{y^2}{(v_0/k)^2} = 1.$$

If the retardation be replaced by acceleration, i.e. the sign of k^2 be changed the orbit will be the hyperbola.

$$\frac{x^2}{a^2} - \frac{y^2}{(v_0/k)^2} = 1.$$

The first case may be approximately realised experimentally, viz. by a mass M , on a smooth horizontal table, attached by an elastic string to a fixed point F and passing through a small smooth hoop at L , the unstretched length of the string being FL .

Ex. 5. P, Q, R are the centres of the squares described externally on the sides of a triangle ABC ; to prove that AP is equal to QR and perpendicular to it.

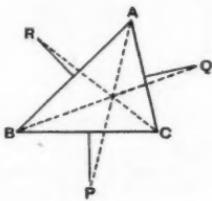


FIG. 6.

Following Hamilton, we use the same letters to denote the vectors of the points from any origin.

$$Q = \frac{C+A}{2} + i \frac{C-A}{2},$$

$$R = \frac{A+B}{2} + i \frac{A-B}{2};$$

$$\therefore Q-R = \frac{C-B}{2} + i \frac{C+B}{2} - iA$$

$$\text{and } P-A = \frac{B+C}{2} + i \frac{B-C}{2} - A;$$

$$\therefore Q-R = i(P-A).$$

This affords the solution of a curious problem, viz.: given the centres of the squares described externally on the sides of a triangle, to construct the triangle.

Ex. 6. If in the above the squares be replaced by equilateral triangles $A'BC$, etc.,

$$A'-A = \frac{B+C}{2} + i \frac{\sqrt{3}}{2}(B-C) - A,$$

$$Q = \frac{C+A}{2} + i \frac{1}{3} \frac{\sqrt{3}}{2}(C-A),$$

$$R = \frac{A+B}{2} + i \frac{1}{3} \frac{\sqrt{3}}{2}(A-B);$$

$$\therefore R-Q = \frac{B-C}{2} + i \frac{1}{3} \frac{\sqrt{3}}{2}(C+B) - i \frac{1}{3} \frac{\sqrt{3}}{2}A;$$

$$\therefore A'-A = \sqrt{3}i(R-Q);$$

and similarly for $B'-B$, $C'-C$.

Now AA' , BB' , CC' are known to be equal;

$\therefore PQR$ is an equilateral triangle.

Turning now to multiplication and division, and using polar coordinates,

$$OP = x_1 + iy_1 = r_1(\cos \theta_1 + i \sin \theta_1) = r_1 \operatorname{cis} \theta_1,$$

$$OQ = x_2 + iy_2 = r_2 \operatorname{cis} \theta_2,$$

$$OP \cdot OQ = r_1 r_2 \operatorname{cis}(\theta_1 + \theta_2),$$

$$OQ/OP = r_2/r_1 \operatorname{cis}(\theta_2 - \theta_1).$$

The product is dependent on the position of the axis; the quotient not. If OA, OB, OC, OD are vectors (lengths r_1, r_2, r_3, r_4) in proportion, so that

$$OD/OC = OB/OA,$$

then, as in de Moivre work, we must have

$$r_4/r_3 = r_2/r_1,$$

and

$$\hat{COD} = \hat{AOB}.$$

Hence the triangles COD, AOB are directly similar.

If we take OA as unit of reference,

$$OD = OB \cdot OC.$$

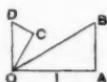


FIG. 7.

Hence to multiply OC by OB we must construct on OC a triangle directly similar to OAB .

In particular, if $OB = \operatorname{cis} \theta$, so that $r_2 = 1$, to multiply OC by OB is to turn it round O through the angle θ —an extension of the rule for i .

A case of proportional vectors occurs in the study of the convex quadrilateral.

Ex. 7. Let the opposite sides AB, CD produced meet in E , and let O be the intersection of the circumcircles DAE, CBE .* Then

$$\hat{OAE} = \hat{ODE};$$

$$\therefore \hat{OAB} = \hat{ODC},$$

$$\hat{OBE} = \hat{OCE};$$

and

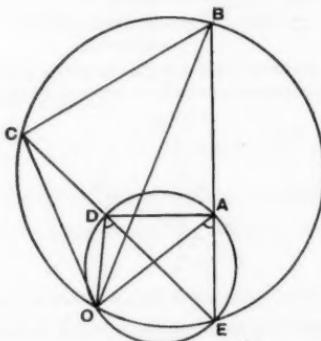


FIG. 8.

\therefore triangles OAB, ODC are directly similar.

* Variously called the Wallace point, Point de Miquel, and focus of the quadrilateral.

Ex. 8. Next let $OB=OC$, or, changing notation, SPT, STQ be similar triangles, so that if

$$SP = 1,$$

$$ST = z,$$

then

$$SQ = w = z^2.$$

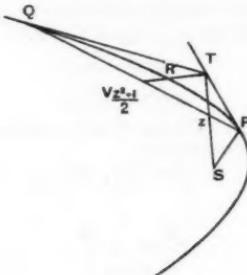


FIG. 9.

Now TP, TQ are tangents to a parabola, focus S , and we see that if T move along a straight line PT , Q describes a parabola.

Let V be the mid-point of PQ .

Then vector $TV = \frac{1}{2}(z^2 + 1) - z = \frac{1}{2}(z - 1)^2 = \frac{1}{2}PT^2$, and is fixed in direction.

Let R be the mid-point of TV .

Then $SR = \frac{1}{2}\left(\frac{z^2+1}{2} + z\right) = \left(\frac{z+1}{2}\right)^2$; but $\frac{z+1}{2}$ gives a point on PT ;

$\therefore R$ is a point on the curve;

$\therefore PT^2 = 2TV = 4TR = 4SP \cdot TR$, since $SP = 1$.

This is a useful form of the familiar $QV^2 = 4SP \cdot PV$.

Ex. 9. The orthocentre of the triangle formed by three tangents (for α, β, γ) is given by $\beta\gamma + \gamma\alpha + \alpha\beta - \alpha\beta\gamma$,

and it is easy to show that this is on the directrix $\rho = 2 + Mi$, or, thus,

$$(1-\alpha)(1-\beta)(1-\gamma) = M(i);$$

$$\therefore \Sigma\beta\gamma - \alpha\beta\gamma = M(i) + \alpha + \beta + \gamma - 1 \\ = M(i) + 2.$$

Next, supposing we had not observed the parabola property, let us consider the case of continued proportion, usually expressed as the transformation

$$w = z^2,$$

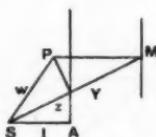


FIG. 10.

and let Y , the end of vector z , move on the line AY at right angles to SA , the unit of reference, $SP=w$. Then all vectors, or operators, represented by segments of AY are multiples of i ;

$\therefore (z-1)^2$ is scalar (i.e. independent of i);

$\therefore z^2 - 2z$ is parallel to AS ,

i.e. if $SM=2SY$ (and $\therefore M$ on a parallel to AY), PM is parallel to SA .

Since

$$S\hat{Y}P = S\hat{A}Y = \text{right angle},$$

$$SP = PM,$$

i.e. locus of P is a parabola.

For the tangent at P we have $dw = 2z dz$, but $dz = \text{multiple of } i$;

\therefore tangent at P is perpendicular to SY , i.e. is PY .

Another method, often useful, is to put $AY = t$, so that

$$z = 1 + ti,$$

$$w = 1 + 2ti - t^2,$$

and the coordinates of P are $1 - t^2, 2t$.

After working with these we can always return to vectors by

$$t = \frac{z-1}{i}.$$

Ex. 10. To find the intersection of the tangents at the points $z = a, \beta$.

Since $\beta - 1 = M(i)$,

$a(\beta - 1)$ is perpendicular to a ;

$\therefore a\beta$ lies on line through and perpendicular to a ,

i.e. on tangent at a^2 .

By symmetry it lies also on tangent at β^2 ;

$\therefore T$, the point of intersection, is given by $a\beta$.

If T, Q be the points of contact, the proportion

$$a^2 : a\beta :: a\beta : \beta^2$$

proves the similarity of the triangles SPT, STQ .

Ex. 11. Again, taking a third tangent from γ , the vertices of the circumscribing triangle are $\beta\gamma, \gamma a, a\beta$.

Now $(a-1)\beta\gamma$ is perpendicular to $\beta\gamma$,

i.e. $a\beta\gamma - \beta\gamma$ " "

\therefore circle on $a\beta\gamma$ as diameter passes through $\beta\gamma$, and similarly $\gamma a, a\beta$.

\therefore circumcircle of the triangle passes through the focus, and its centre is given by $\frac{a\beta\gamma}{2}$.

Ex. 12. Taking four tangents,

$$\frac{a\beta\gamma\delta}{2} (\delta - 1) \text{ is perpendicular to } \frac{a\beta\gamma}{2}, \text{ etc.}$$

\therefore circle on $\frac{a\beta\gamma\delta}{2}$ as diameter passes through the circum-centres of the four triangles formed by the tangents, and its centre is the point whose vector is $\frac{a\beta\gamma\delta}{4}$.

This reasoning may be continued to obtain Miquel's theorem.

Ex. 13. The intersections of a fifth tangent with the preceding are

$$\epsilon a, \epsilon\beta, \epsilon\gamma, \epsilon\delta.$$

Their vectors are those of a, β, γ, δ turned through the same angle.
We have thus congruent pencils;

\therefore cross ratios are constant.

Ex. 14. Again the vectors of the joins of the points of contact are
 $\epsilon^2 - \alpha^2, \quad \epsilon^2 - \beta^2, \quad \epsilon^2 - \gamma^2, \quad \epsilon^2 - \delta^2,$
and are therefore perpendicular to

$$\epsilon + \alpha, \quad \epsilon + \beta, \quad \epsilon + \gamma, \quad \epsilon + \delta,$$

and these have the same cross ratios as the pencil $\alpha, \beta, \gamma, \delta$.

The case of four tangents merits special attention, since they determine a parabola uniquely, and may be the sides of any given quadrilateral.



FIG. 11.

Ex. 15. If M, N are mid-points of diagonals,

$$M = \frac{1}{2}(\alpha\gamma + \beta\delta),$$

$$N = \frac{1}{2}(\beta\gamma + \alpha\delta);$$

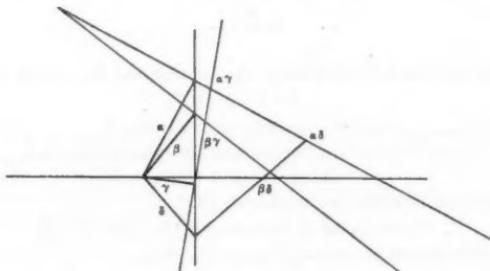


FIG. 12.

$$\therefore M - N = \frac{1}{2}(\alpha - \beta)(\gamma - \delta) = M(i) \cdot M(i) = \text{scalar};$$

$\therefore MN$ is parallel to axis.

Similarly join of M to mid-point of third diagonal is parallel to axis.

Ex. 16. Again, product of two sides meeting in $\alpha\gamma$ is $(\alpha\delta - \alpha\gamma)(\beta\gamma - \alpha\gamma)$

$$= \alpha\gamma(\delta - \gamma)(\beta - \alpha)$$

$$= \alpha\gamma \cdot 2MN,$$

and similarly for other pairs of sides.

$\therefore 2MN$ = product of two adjacent sides divided by the focal distance of their common vertex, a result first obtained, otherwise, by Mr. R. F. Davis, M.A.

Ex. 17. Let G_1, G_2 , etc., be the C.'s of G. of the triangles corresponding to $\beta, \gamma, \delta; \gamma, \delta, \alpha$, etc., and P_1, P_2 , etc., the points of contact of the tangents. Then the sides and diagonals of the quadrangle G_1, G_2, G_3, G_4 are parallel to those of $P_1P_2P_3P_4$.

For

$$G_2 - G_1 = \frac{1}{2}(\gamma\delta + \delta\alpha + \alpha\gamma) - \frac{1}{2}(\beta\gamma + \gamma\delta + \delta\beta)$$

$$= \frac{1}{2}(\alpha - \beta)(\delta + \gamma)$$

$$= \frac{1}{2} \frac{\alpha - \beta}{\delta - \gamma} (\delta^2 - \gamma^2),$$

but $\frac{\alpha - \beta}{\delta - \gamma}$ is scalar;

$\therefore G_1G_2$ is parallel to P_3P_4 , etc.

But the quadrangles are not similar.

There is a reminder of what may be called the "zig-zag" theorem.

$ABCD$ is any quadrilateral, CE, DF parallel to AC, BD meeting AD, BC in E, F ; then EF is parallel to AB .

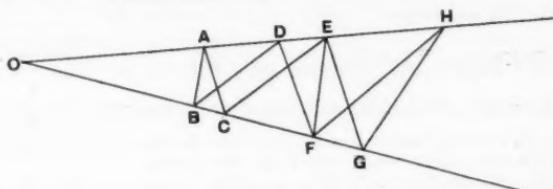


FIG. 13.

If DA, CB meet in O ,

$$\triangle OAF = \triangle OCD = \triangle OBE, \text{ etc.}$$

In his *Foundations of Geometry*, Hilbert gives a tortuous proof, bringing in a circle!

Ex. 18. Since the normal at z^2 is parallel to z , we have for the intersection of two normals $N = a^2 - pa = \beta^2 - q\beta$.

Putting $a = 1 + t_1 i$, $\beta = 1 + t_2 i$, equating reals, etc., we obtain two simple equations for p and q ; hence

$$N = a\beta(a + \beta - 3).$$

If the tangents meet in $T = a\beta$,

$$\frac{N+T}{2} = a\beta(a - 1 + \beta - 1),$$

or,

$$SM = a\beta M(i) \\ = \text{perpendicular to } ST,$$

M being the mid-point of NT .

Cor. Putting $\beta = a$, N is the centre of curvature, T on curve

$$N = 2a^3 - 3a^2.$$

The theorem about M explains why the focal chord of curvature at $T = 4ST$.

A word may be added in support of the method of operator i (as against $\sqrt{-1}$), viz. $a\beta\gamma \dots (1)$ is intelligible, all lines being formed from the unit of reference (1), whereas $a \times \beta \times \gamma \times \dots$

as pointed out by Marcolongo and others, is confusing. Why should the product of two vectors be another vector? The only serviceable interpretation, is that suggested to Grassmann by his father, to take it as a new compound unit, the area moved through by a when translated parallel to β . The product turns out to be the analytical equivalent of a couple in Statics.

R. W. GENÈSE.

GLEANINGS FAR AND NEAR.

166. Speaking of Abel's famous essay on the Quintic the dean says, "The clearest mind gets bewildered by the extreme complexity of the relations which it is necessary to consider in such investigations, and our *final assent* to the conclusion reached is rather a *tacit acquiescence* in reasonings, whose entire relevancy we can neither fully appreciate nor easily refute, than the spontaneous admission of a truth the evidence for which is complete and irresistible."—Peacock's *Algebra*, Part II., 2nd ed., *per* R. W. Genèse.

THE CONIC IN PARAMETERS.

1. *Conic.* The equations

$$\frac{x}{a_1t^2+2b_1t+c_1} = \frac{y}{a_2t^2+2b_2t+c_2} = \frac{z}{a_3t^2+2b_3t+c_3} = \frac{1}{a_4t^2+2b_4t+c_4} \quad \dots \dots \dots (1)$$

represent a conic; for putting each equal to $\frac{1}{\lambda}$ and eliminating λ, t^2, t , the curve lies in the plane in the margin, while any other plane cuts it in two points.

The curve cuts the line at infinity in points whose parameters are given by $a_4t^2+2b_4t+c_4=0$(2)
Hence the condition the conic should be an ellipse, parabola, or hyperbola is $a_4c_4 - b_4^2 >, =, \text{ or } < 0$.

Secant. The equations

$$\frac{x}{a_1t^2+2b_1t+c_1 - a_1(t-t_1)(t-t_2)} = 3 \text{ similar expressions} \quad \dots \dots \dots (3)$$

are linear in t , and therefore represent a straight line which, moreover, passes through the points on the conic whose parameters are t_1 and t_2 .

Tangent; Asymptotes. Putting $t_2=t_1$ and writing t instead of $2t-t_1$, the tangent at t_1 is given by the equations

$$\frac{x}{a_1tt_1+b_1(t+t_1)+c_1} = \text{etc.}, \quad \dots \dots \dots (4)$$

and if t_1 is a (real) root of (2) it is an asymptote.

Pole of Chord. Also the pole of the chord t_1, t_2 is

$$\frac{x}{a_1t_1t_2+b_1(t_1+t_2)+c_1} = \text{etc.}, \quad \dots \dots \dots (5)$$

since it lies on the tangents at t_1 and t_2 .

Centre. When t_1 and t_2 are the roots of (2), the pole of the line at infinity, i.e. the centre, is

$$\frac{x}{a_4c_1+a_1c_4-2b_1b_4} = \text{etc.} = \frac{1}{2(a_4c_4 - b_4^2)}. \quad \dots \dots \dots (6)$$

Diameter. Since the pole of a diameter lies on the line at infinity, the parameters of its extremities satisfy the equation

$$a_4t_1t_2+b_4(t_1+t_2)+c_4=0. \quad \dots \dots \dots (7)$$

2. In the figure, if CP and CD are conjugate diameters, P is the middle point of QR , so that

$$(x_2-x_1)+(x_3-x_1)=0. \quad \dots \dots \dots (8)$$

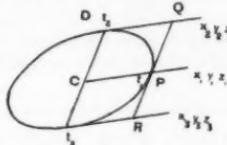


FIG. 1.

Now

$$\begin{aligned} x_2 - x_1 &= \frac{t_2 p + q}{t_2 p' + q'} - \frac{t_1 p + q}{t_1 p' + q'} \\ &= \frac{(t_2 - t_1)(pq' - p'q)}{(t_2 p' + q')(t_1 p' + q')}, \end{aligned} \quad \dots \dots \dots (9)$$

where $p = a_1t_1 + b_1$, $q = b_1t_1 + c_1$, $p' = a_4t_1 + b_4$, $q' = b_4t_1 + c_4$.

Hence equation (8) becomes

$$\frac{t_2 - t_1}{a_4 t_1 t_2 + b_4 (t_1 + t_2) + c_4} + \frac{t_3 - t_1}{a_4 t_1 t_3 + b_4 (t_1 + t_3) + c_4} = 0,$$

or using condition (7) for t_2 and t_3 ,

$$a_4 t_1 t_2 + b_4 (t_1 + t_2) + c_4 = \sqrt{a_4 c_4 - b_4^2} (t_1 - t_2). \quad \dots \dots \dots (10)$$

Conjugate Diameters. Writing $a_4 t_1 + b_4 = \sqrt{a_4 c_4 - b_4^2} \tan \frac{\alpha}{2}$, and similarly for t_2 and β , this becomes $\alpha - \beta = \frac{\pi}{2}$.

For the equiconjugates $\alpha = \frac{\pi}{4}$, and the four extremities are given by

$$a_4 t + b_4 = \pm (\sqrt{2 \pm 1}) \sqrt{a_4 c_4 - b_4^2}.$$

For the vertices DQP must be a right angle, and from the figure

$$\Sigma((x_2 - x_3)^2 + (x_3 - x_1)^2 - (x_1 - x_2)^2) = 0,$$

or

$$\Sigma(x_2 - x_1)(x_3 - x_1) = 0. \quad \dots \dots \dots (11)$$

In equation (9)

$$\begin{aligned} pq' - p'q &= (a_1 t + b_1)(b_4 t + c_4) - (a_4 t + b_4)(b_1 t + c_1) \\ &= (b_1 c_4 - b_4 c_1) + t(a_1 c_4 - a_4 c_1) + t^2(a_1 b_4 - a_4 b_1) \\ &= \begin{vmatrix} 1, & -t_1, & t^2 \\ a_1, & b_1, & c_1 \\ a_4, & b_4, & c_4 \end{vmatrix}. \end{aligned}$$

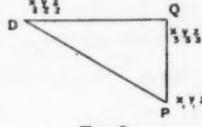


FIG. 2.

Vertices. Substituting in equation (11), the conditions that t_1 and t_2 should be vertices reduce to

$$\sum_{\tau}^3 \begin{vmatrix} 1, & -t_1, & t_1^2 \\ a_\tau, & b_\tau, & c_\tau \\ a_4, & b_4, & c_4 \end{vmatrix} \times \begin{vmatrix} 1, & -t_2, & t_2^2 \\ a_\tau, & b_\tau, & c_\tau \\ a_4, & b_4, & c_4 \end{vmatrix} = 0, \quad \dots \dots \dots (12)$$

together with equation (10).

3. *Examples.* In what follows capital letters denote minors of small letters in the determinant

$$\Delta \equiv \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}.$$

Also $E \equiv a_1^2 + a_2^2 + a_3^2$, $F \equiv a_1 b_1 + a_2 b_2 + a_3 b_3$, $G \equiv b_1^2 + b_2^2 + b_3^2$.

(i) The parabola $x = a_1 t^2 + 2b_1 t + c_1$, $y = a_2 t^2 + 2b_2 t + c_2$, $z = a_3 t^2 + 2b_3 t + c_3$ lies in the plane $C_1 x + C_2 y + C_3 z = \Delta$.

Where any plane $lx + my + nz + p = 0$ cuts it

$$t^2(a_1 l + a_2 m + a_3 n) + t(-) + \dots = 0,$$

and if one value of t is infinite $a_1 l + a_2 m + a_3 n = 0$.

Also $a_1 C_1 + a_2 C_2 + a_3 C_3 \equiv 0$. Hence the line whose direction cosines are as $a_1 : a_2 : a_3$ is parallel to this plane and the plane of the conic, and is therefore parallel to the axis.

The direction cosines of a tangent are as $\dot{x} : \dot{y} : \dot{z}$, i.e. as

$$(a_1 t + b_1) : (a_2 t + b_2) : (a_3 t + b_3).$$

Hence the parameter of the vertex is given by $\Sigma a_i (a_i t + b_i) = 0$ or by $Et + F = 0$.

The tangents at the *extremities of the latus rectum* make angles of $\pm 45^\circ$ with the axis; hence in this case

$$\frac{\Sigma a_i(a_it+b_i)}{\sqrt{(\Sigma a_i^2)(\Sigma(a_it+b_i)^2)}} = \pm \frac{1}{\sqrt{2}},$$

i.e.

$$2(Et+F)^2 = E(Et^2 + 2Ft + G),$$

or

$$(Et+F)^2 = EG - F^2;$$

whence at once the *latus rectum* is

$$4(EG - F^2)/E^{\frac{3}{2}} \quad \text{or} \quad 4(C_1^2 + C_2^2 + C_3^2)/(a_1^2 + a_2^2 + a_3^2)^{\frac{3}{2}},$$

and the *x* coordinate of the *focus* is $(a_1G - 2b_1F + c_1E)/E$.

Tangents at t_1, t_2 will be at right angles when

$$\Sigma(a_1t_1+b_1)(a_1t_2+b_1) = 0, \quad \text{i.e. when } Et_1t_2 + F(t_1+t_2) + G = 0.$$

Hence putting $t_1t_2 = \lambda, t_1+t_2 = \mu$ in this and equation (4), the *directrix* is given by

$$x = a_1\lambda + b_1\mu + c_1, \quad y = a_2\lambda + b_2\mu + c_2, \quad z = a_3\lambda + b_3\mu + c_3, \quad 0 = E\lambda + E\mu + G.$$

(ii) The *hyperbola* $x = a_1\lambda + \frac{b_1}{\lambda} + c_1$, etc.

For all values of λ , the mean of the coordinates of the points whose parameters are $\pm \lambda$ are (c_1, c_2, c_3) . Hence this is the *centre*.

$$\begin{aligned} \text{Again } (x - c_1)^2 + (y - c_2)^2 + (z - c_3)^2 &= \Sigma \left(a_i \lambda + \frac{b_i}{\lambda} \right)^2 \\ &= E\lambda^2 + 2F + G/\lambda^2 \\ &= (\sqrt{E} \cdot \lambda \pm \sqrt{G/\lambda})^2 + 2(F \mp \sqrt{EG}). \end{aligned}$$

Hence the squares of the *semiaxes* are $2(F \mp \sqrt{EG})$ and the parameters of the *vertices* are given by $\lambda^4 = G/E$.

Proceeding as for the axis of the parabola in Ex. 1, the directions of the *asymptotes* are as $a_1 : a_2 : a_3$ or $b_1 : b_2 : b_3$; and their equations are

$$\frac{x - c_1}{a_1} = \frac{y - c_2}{a_2} = \frac{z - c_3}{a_3} \quad \text{and} \quad \frac{x - c_1}{b_1} = \frac{y - c_2}{b_2} = \frac{z - c_3}{b_3},$$

while the condition for a rect. hyp. is Σab or $F = 0$.

(iii) The *ellipse* $x = (a_1 + b_1t)/(1 + t^2)$, etc.

The *centre* is $(a_1/2, a_2/2, a_3/2)$.

$$\begin{aligned} \text{Also } \frac{a+bt}{1+t^2} - \frac{a}{2} &= \frac{a(1-t^2)+2bt}{2(1+t^2)} \\ &= \frac{1}{2}(a \cos \theta + b \sin \theta), \text{ where } t = \tan \frac{\theta}{2}. \end{aligned}$$

Hence the square of a central radius vector is

$$\begin{aligned} &\frac{1}{8}(E \cos^2 \theta + 2F \cos \theta \sin \theta + G \sin^2 \theta) \\ &= \frac{1}{8}\{(E+G) + (E-G) \cos 2\theta + 2F \sin 2\theta\} \\ &= \frac{1}{8}\{(E+G) + \sqrt{(E-G)^2 + 4F^2} \cos(2\theta - \alpha)\}, \text{ where } \tan \alpha = \frac{2F}{E-G}. \end{aligned}$$

Hence the squares of the *semi-axes* are

$$\frac{1}{8}\{(E+G) \pm \sqrt{(E-G)^2 + 4F^2}\},$$

and the area is $\pi \sqrt{EG - F^2}/4$.

The *vertices* are given by $2\theta = a$ or $\pi + a$, i.e. by $\tan 2\theta = \tan a$, or by

$$\frac{4t(1-t^2)}{1-6t^2+t^4} = \frac{2F}{E-G}.$$

A NOTE ON TUCKER'S HARMONIC QUADRILATERAL.

In a previous note it has been pointed out as a corollary to the proof of Ptolemy's Theorem that, if in a cyclic quadrilateral $ABCD$, $AB \cdot CD = BC \cdot DA$, and if M be the mid-point of the diagonal BD ($2m$),

$$MA \cdot MC = MB^2$$

and

$$\hat{BMA} = \hat{CDA} = \hat{CMB},$$

so that A and C are each the image in DB of the inverse of the other w.r.t. M , the constant of inversion being m^2 .

In what follows the combined operation of (1) inverting with respect to M with constant m^2 , and (2) taking the image of the inverse in DB will be called the *vectorial inversion* (M, m^2, DB).*

Conversely, if any two points be such that each could be obtained from the other by this "vectorial inversion," they will lie on a circle through B, D (i)

A quadrilateral such as the above (sometimes called a harmonic quadrilateral) has some remarkable properties discovered by Mr. R. Tucker, who demonstrated them mainly by Trigonometrical and Algebraical work. I propose to show how they follow through vectorial inversion and other geometrical transformations.

1. Let E be the intersection of the diagonals of such a quadrilateral, O the centre of its circumcircle; then two points P, P' can be found on the circle on OE as diameter such that

$$\hat{PAB} = \hat{PBC} = \hat{PCD} = \hat{PDA}$$

and

$$\hat{P'BA} = \hat{P'CB} = \hat{P'DC} = \hat{P'AD}.$$

Since $\hat{AMB} = \hat{BMC}$, $\therefore OM, CA$ meet at Y the pole of BD .

Similarly, if L be mid-point of AC ($= 2l$), OL, DB meet at Z the pole of AC .

Also, if AB, CD meet in I and BC, DA in J , I, Y, J, Z are collinear and form a harmonic range.

Hence $ME \cdot MZ = MB^2 = MY \cdot MO$.

\therefore the vectorial inverse (M, m^2, DB) of $IYZJ$ is $\odot MEO$, having OE ($= 2\rho$) as diameter (ii)

Also, since the conjugate rays MY, MZ of the harmonic pencil $M(IYZJ)$ are at right angles, they are the bisectors of \hat{IMJ} .

Produce IM, JM to meet $\odot MEO$ in P, P' (so that P, P' are vectorial inverses of J, I), then angles IMD, DMP', JMB, BMP are all equal. Let each equal ϕ .

JBC inverts into $\odot PBAM$ touching BC at B , (iii)

JAD , , $\odot PCDM$, , AD at D (iv)

Then

$$\hat{PAB} = \hat{PBC}, \text{ by (iii),}$$

$$= \hat{PDA} \quad (\because J, B, P, D \text{ are concyclic})$$

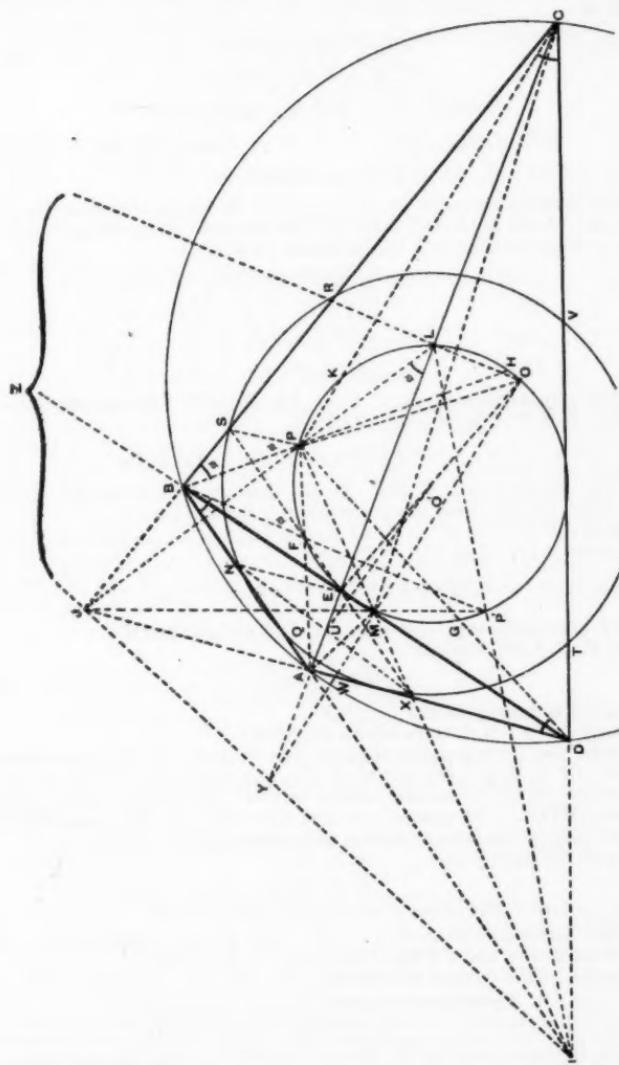
$$= \hat{PCD}, \text{ by (iv).}$$

Similarly with the same inversion (M, m^2, DB)

(since IAB inverts into $\odot P'CBM$ touching AB at B , (v)

ICD , , $\odot P'ADM$, , CD at D , (vi)

* See M. Ant. Gob, *Sur une série de quadrangles*. Congrès de Marseilles, 1891.



I, D, P', B are concyclic), angles $P'BA, P'AD, P'DC, P'CB$ are each equal to ϕ .

From the above properties P and P' have been called the Brocard points of $ABCD$.

Again,

$$\hat{P}LE = \hat{P}ME = \phi = \hat{P}BC;$$

$\therefore P, B, L, C$ are concyclic;

\therefore angles BCA, BPL are supplementary.

But

$$\hat{B}CA = \hat{B}DA = \hat{B}DJ \quad (\because J, P, B, D \text{ are concyclic});$$

$\therefore JPL$ is a straight line.

\therefore the vectorial inversion (L, l^2, CA) of $IYJZ$ would determine the same points on $\odot MEO$ as (M, m^2, DB) , I, J now respectively inverting into P, P' .

It is to be noted that with this inversion (L, l^2, CA)

$$\left. \begin{array}{lll} IAB \text{ inverts into } \odot PADL \text{ touching } AB \text{ at } A, \\ IDC \quad " \quad \odot PBCL \quad " \quad CD \quad " \quad C, \\ JBC \quad " \quad \odot P'DCL \quad " \quad BC \quad " \quad C, \\ JAD \quad " \quad \odot P'ABL \quad " \quad DA \quad " \quad A \end{array} \right\} \dots \dots \dots \text{(vii)}$$

II. The intersections F, H, K, G of PA, PB, PC, PD respectively with $P'B, P'C, P'D, P'A$ lie on the $\odot MEO$.

For

$$\hat{P}FB = 2\phi = \hat{P}OP', \text{ and so forth.} \dots \dots \dots \text{(viii)}$$

III. The parallels to AB, BC, CD, DA through E meet the sides in eight points which lie on a circle concentric with $\odot MEO$.

Produce EF to meet BC in S and AD in X .

Bisect OE in O' . Join $O'P, O'P', PS$.

The

$$\hat{P}FS = \hat{E}FP' = \hat{E}OP' = \phi = \hat{P}BC;$$

$\therefore EF$ bisects the angles between $PA, P'B$ and $PBS = PFS$.

$\therefore P, F, B, S$ are concyclic;

$$\therefore \hat{S}PB = \hat{S}FB = \phi. \dots \dots \dots \text{(ix)}$$

$\therefore \triangle PBS$ is directly similar to $\triangle POO'$.

Similarly $\triangle P'AX$ is directly similar to $\triangle P'OO'$.

Similarly, also, if the parallels through E to BC, CD, DA meet the adjacent sides CD, AB in V, Q ; BC, DA in R, W ; CD, AB in T, N ; the $\triangle s$ PCV, PDW, PAN will all be directly similar to POO' , and the $\triangle s$ $P'DT, P'CR, P'BQ$ to $\triangle P'OO'$. (Several of the lines referred to in this paragraph are indicated only by the letters marking their extremities.)

Hence all the eight points

$S, V, W, N,$

$X, T, R, Q,$

lie on a \odot centre O' and radius ρ' such that $\rho' : R = O'P : OP$. $\dots \dots \dots \text{(x)}$

IV. Also quadrilaterals $NSVW, ABCD$ are directly similar with P for centre of similitude, and $XTRQ, ABCD$ with P' for centre.

And since $\triangle EPP'$ is similar to $O'PO$,

E corresponds in $NSVW$ to P' in $ABCD$

and

$" \quad " \quad XTRQ$ to P in $ABCD$. $\dots \dots \dots \text{(xi)}$

So that, as pointed out by Mr. Tucker, P and E are the Brocard points of $NSVW$, E and P' those of $XTRQ$.

We can, however, obtain these results by a different transformation of $ABCD$.

NOTE ON TUCKER'S HARMONIC QUADRILATERAL. 309

Since $ENAX$ is a parallelogram, its diagonals EA, NX bisect each other. Let them cut at U . Since $EO' = \frac{1}{2}EO$,

$$\therefore O'U = \frac{1}{2}OA, \text{ and is also parallel to it.}$$

\therefore it is perpendicular to XN , since AM, AE are isogonals.

Again, since $\triangle s DXE, XEN, ENB$ are similar,

$$DE : NX = EX : EN = NX : EB;$$

$$\therefore NX^2 = DE \cdot EB = R^2 - 4\rho^2.$$

Similarly for QS, WT, MR .

$\therefore N, S, V, W; X, T, R, Q$ all lie on a \odot centre O' and radius ρ' , where $\rho' = O'N$

$$(and \quad \therefore \rho'^2 = \frac{1}{4}R^2 + \frac{1}{4}R^2 - \rho^2,$$

$$\rho'^2 + \rho^2 = \frac{1}{2}R^2).$$

Also, by (ix), $PSR = 2\phi$, and similarly $PNB = 2\phi$.

$\therefore N$ lies on $\odot PFBS$;

$$\therefore \hat{NSX} = \hat{NPE} = \phi;$$

$$\therefore \hat{XO'N} = 2\phi = \hat{EOP}.$$

\therefore quadrilateral $NSVW$ can be obtained from $ABCD$ by (1) an "H" transformation ($E, \frac{1}{2}$) A to U , and so on, followed by (2) an " HR " ($O', O'N/OV, \bar{\phi}$). But these leave P unchanged, i.e. it is a double point of the compound transformation, while P' is transformed into E , as in (x). Similarly (xi) follows.

V. It is easily seen that the feet of perpendiculars from P, P' to AB, BC, CD, DA lie on a \odot whose centre is the mid-point of PP' , and hence that P, P' are foci of an ellipse inscribed in $ABCD$.

VI. E is the centre of mean position of its projections on the sides.

For let $Ea, E\beta, E\gamma, E\delta$ (not shown in the diagram) be perpendicular to AB, BC, CD, DA , and let a, β, γ, δ be projected on DB .

$\therefore AE$ is isogonal with the median AM of $\triangle DAB$,

$$\therefore E\delta : Ea = DA : AB \quad \}$$

and

$$DE : EB = DA^2 : AB^2; \quad \}$$

$$\therefore E\delta : Ea^2 = DE : EB;$$

$\therefore E\delta, Ea$ have equal projections on DB ;

$\therefore E$ is projection of mid-point of δa on DB .

Similarly $E \quad \beta \quad \gamma \quad \delta$

\therefore straight line joining mid-points of $\delta a, \beta\gamma$ passes through E ;

\therefore similarly $a \quad \beta \quad \gamma \quad \delta$

$\therefore E$ is centre of mean position of a, β, γ, δ .

EDWARD M. LANGLEY.

167. Many years ago the late Dr. Hayward invited the Council of the A.I.G.T. to meet at Harrow, giving generous hospitality. At lunch the conversation turned on Euclid's test of proportionals, and our host defended the system of multiples, and added, "We should have a good idea of the relative contents of this wine glass and that tumbler if we counted the number of times they went, respectively, into a barrel." Whereupon Dr. Sophie Bryant replied, "Wouldn't it do equally well if we counted the numbers of times a thimbleful of water would go into each?" —Per R. W. Genese.

MATHEMATICAL NOTES.

657. [K¹. 8. b.] *Addition to Note 654*, vol. xi. p. 237.

Cor. If $DX = XB$, the quadrilaterals $ABCD$, $ABC'D$ have properties worth noting.

(i) In $ABCD$, $\triangle ADC = \triangle ABC$;

$$\therefore AD \cdot DC = AB \cdot BC \text{ (ADC, ABC supply.)};$$

$$\therefore DA : AB = BC : CD,$$

i.e. the sides taken in the order DA, AB, BC, CD are proportionals.

(ii) In $ABC'D$, $\because AD \cdot DC = AB \cdot BC$,

$$\therefore AD \cdot BC' = AB \cdot C'D,$$

i.e. the rectangles contained by pairs of opposite sides are equal.

Again, since $C'X, AX$ are equally inclined to AD at its mid-point X , $C'A$ passes through the pole of BD . Similarly DB passes through the pole of AC' . Such a quadrilateral is sometimes called a "harmonic quadrilateral." It has further remarkable properties discovered by Mr. R. Tucker.

E. M. LANGLEY.

658. [V. 7.] *Note on Gleanings Far and Near*, 138, *Gazette*, xi. p. 143.

De Morgan's statement has sent me back to J. de Witt's *Elementa Linearum Curvarum*—a work I have known for many years past. The terms *crus patiens* and *crus efficiens* are certainly used by de Witt, but not in the sense indicated by De Morgan. De Witt is dealing in his opening pages with an angle, one side of which is fixed. This "leg" he calls, naturally enough, *crus patiens*. The other, which is not fixed, is his *crus efficiens*.

I send this Note because incorrect historical references now current are sufficiently numerous to justify our keeping a watchful eye upon affirmations made even by one whose authority is usually unimpeachable.

Geneva, Nov. 22, 1922.

GINO LORIA.

659. [B. 12. d.] Can any reader tell me anything of the history of the word "vector" in its transition from the radius vector of the astronomers to the meaning we associate with Hamilton? The classical account of the word is on p. 15 of the *Lectures on Quaternions* (1853), but in a paper read to the Royal Irish Academy in 1843 (*Transactions*, vol. 21) occurs the following sentence (p. 274):

"We may also say that i is the *imaginary unit*, or perhaps, more expressively, that it is the *VECTOR UNIT*, of the same quaternion."

And this without apology or explanation is the earliest use of the word by Hamilton that I have myself come across. How was it that his audience knew what he meant?

There are no references in the *Encyk. d. Math. Wiss.*, in Graves' *Life of Hamilton*, or in the *N.E.D.*, that throw any light on the matter, and Dr. Craigie has ransacked his pigeon-holes in vain to help me.

University College, Reading.

E. H. NEVILLE.

660. [K¹. 1. c.] *Morley's Theorem* (v. Note 621).

In the figure, *Gazette*, vol. xi. p. 85, let BRL cut AQ in U ; AQ produced cuts BP in N and CP in V ; CP cuts AR in M ; QM cuts RN in O .

Then BP, BL are isogonal, and so are CP, CL ;

$$\therefore AP, AL \text{ are also isogonal} ;$$

$$\therefore A(BRLU) = A(CVPM) ;$$

$$\therefore N(BRLU) = Q(CVPM) = Q(PMCV),$$

and these pencils have a common ray; \therefore their corresponding rays have collinear intersections, i.e. P, O, L are collinear.

As R is the in-centre of ANB , $\hat{ARN} = 90^\circ + \frac{1}{2}B$.

As Q is the in-centre of AMC , $\hat{RMQ} = 90^\circ - \frac{1}{2}A - \frac{1}{2}C$;

∴ the difference, viz., $\hat{R}OM = 60^\circ$.

Similarly the other angles at O are 60° ; since they have a common base and equal angles at each of its extremities, the triangles ORL, OQL are congruent, and so are the triangles PRL, POL .

The College, Marlborough.

A. ROBSON.

661. [§¹, 21, 4.] *Squaring the Circle.*

The following solution of the problem of squaring the circle is due to my friend, Mr. T. C. Dennison. It is accurate enough to satisfy the needs of any draughtsman.

Let AB be the diameter of a circle. Join B to M , the middle point of the quadrant AC .

Let P be the point on BM such that AP is equal to AO , the radius of the circle. Let AP produced cut the circle at Q .

$$4Q = 4Q[2 - \sqrt{2} + \frac{1}{2}] \quad \text{and that} \quad 2 - \sqrt{2} + \frac{1}{2} = 1.7750$$

and therefore differs from $\sqrt{\pi}$, which is 1.7725, by only 14 parts in 1000.

This numerical relation seems to be a mere coincidence. Is it more than that?

F. J. W. WHIPPLE.

662. [D. 2. b.] *A Problem upon $\Sigma 1/n$.*

From the well-known divergent series (a),

other series are derived, either (β) by the omission of all fractions in whose denominators a chosen digit (for example 6) occurs, or (γ) by the omission of all fractions in whose denominators a chosen sequence of digits (for example 4739) occurs. Prove that both (β) and (γ) are convergent series.

PROOF.

(a) In the series (a) group together those fractions whose denominators contain the same number of digits. There are

9 of one digit, all of them greater than 1/10;	
90 of two digits,	" " "
900 of three digits,	" " "
9×10^r of $r+1$ digits,	" " "

Hence the sum of the fractions in each group exceeds $9/10$, and by taking a sufficiency of groups we can ensure that the sum of the series (a) shall exceed any chosen number. The series (a) is *divergent*.

(b) Group the terms in (β) in the same way. There are in (β)

8 fractions whose denominators contain one digit ;
8×9 " " " " two digits ;
8×9^2 " " " " three "
8×9^3 " " " " $r+1$ "

This is true because (supposing 6 to be the digit that is barred) the first digit may be any of the eight 1, 2, 3, 4, 5, 7, 8, 9; and the second, third, and

any later digit may be any of the nine 0, 1, 2, 3, 4, 5, 7, 8, 9. Now no fraction with $r+1$ digits in its denominator is greater than $1/10^r$. Hence the sum of the fractions in (β) whose denominators contain $r+1$ digits is less than $8 \times 9^r \times 1/10^r$, and the sum of any number of groups of terms of (β) is less than the sum of the same number of terms of the geometrical progression

$$8 + 8 \times (9/10) + 8 \times (9/10)^2 + 8 \times (9/10)^3 + \dots,$$

i.e. (β) converges to a sum which is certainly less than 80.

A modification is necessary if the digit omitted is 0. The first digit, like all later ones, may then be any of the nine, 1, 2, 3, 4, 5, 6, 7, 8, 9; the 8's in the above results must be replaced by 9's.

(c) The same reasoning may obviously be applied to any scale of notation other than that of 10. When the radix of the scale is s there are s "digits," 0, 1, 2, 3, ... $s-1$. The terms of the series (α) (the reciprocals of the natural numbers) fall into groups containing the same number of "digits," and the sum of the fractions in any group exceeds $(s-1)/s$. Again, in the series (β) , derived by the omission of all fractions in whose denominators a chosen "digit" occurs, the sum of the terms in the r th group is less than the r th term of a geometrical progression with common ratio $(s-1)/s$ and first term ordinarily $s-2$ or exceptionally $s-1$. Therefore in the scale of notation with radix s , the series (β) always converges—to a sum which is certainly less than $s(s-1)$.

(d) In order to prove that the series (γ) is convergent, where all fractions are omitted from (α) in whose denominators a certain sequence such as 4739 of four digits (in ordinary notation) occurs, take $s=10,000$. With this radix the "digits" are the natural numbers up to 9,999, and any number such as 1,702,364,895,701,627,735 is represented as

$$(170)s^4 + (2364)s^3 + (8957)s^2 + (0162)s + (7735),$$

the quantities in brackets being "digits." By the arguments of paragraphs (b) and (c), if all fractions in which the "digit" (4739) occurs are omitted, the resulting series is convergent, and this imposes a more stringent condition than (γ) , since it implies in the ordinary notation that the sequence 4739 must be followed by four, or eight, or twelve, or sixteen ... figures. *A fortiori* the series (γ) must be convergent. In the same way, by taking $s=10^n$, we prove the theorem (γ) for a sequence of n digits. Q.E.D.

663. [Q. 1. a.] *On Note 613, xi. p. 57.*

The type of closed Euclidean space of two dimensions described by Captain Elliott (*Math. Gaz.* xi. 57) is the well-known Clifford's surface in elliptic space (W. K. Clifford, "Preliminary Sketch of Bi-Quaternions," *Proc. London Math. Soc.* 4 (1873), 381-395, or *Math. Papers*, p. 193). For a representation of this surface on the anchor-ring in Euclidean space see the present writer's *Non-Euclidean Geometry* (London, Bell, 1914), p. 106.

Victoria Univ. College, Wellington, N.Z.

D. M. Y. SOMMERVILLE.

664. [V. 6.] *Note on Gleanings*, 88, x. p. 368.

The German Cocker. "Nach Adam Riese." Adam Riese or Ryse (1492-1559), nearly 150 years before Cocker, was the author of widely used books on arithmetic.

D. M. Y. SOMMERVILLE.

665. (V. 1. a. μ.) *On Note 617, xi. p. 59.*

This proof is set as an exercise in Charles Davison's *Subjects for Mathematical Essays* (Macmillan, 1915), p. 73.

Perhaps Mr. Davison (King Edward's High School, Birmingham) could give an earlier reference.

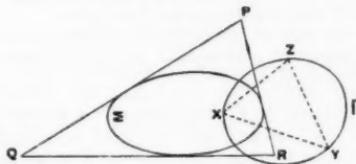
D. M. Y. SOMMERVILLE.

666. [L². 3. r.] On a Tetrahedron autopolar w.r.t. a Conicoid.

It is known that if a tetrahedron is self-polar w.r.t. a conicoid, then the circumscribing sphere of the tetrahedron is orthogonal to the director sphere of the conicoid.

Now if the conicoid is a hyperboloid whose asymptotic cone possesses three mutually perpendicular tangent planes, its director sphere reduces to a point-sphere, viz. the centre of the hyperboloid. In this case, therefore, the centre of the hyperboloid lies on the sphere circumscribing the self-polar tetrahedron.

The following is a geometrical proof of this.



Let Σ be the conic in which the asymptotic cone of the hyperboloid cuts the plane at infinity, and let Γ be the circle at infinity. Let three mutually perpendicular tangent planes to the asymptotic cone meet the plane at infinity in the triangle PQR .

Then PQR circumscribes Σ and is also a self-polar triangle w.r.t. Γ . Hence a triangle XYZ can be inscribed in Γ , self-polar w.r.t. Σ .

Let the centre of the hyperboloid be O .

Then XYZ is a tetrahedron self-polar w.r.t. the hyperboloid.

Hence, if $ABCD$ is any other tetrahedron self-polar w.r.t. the hyperboloid, any conicoid through seven of the eight vertices $ABCDXYZ$ must go through the remaining vertex.

Now the sphere circumscribing $ABCD$ goes through the circle at infinity, and therefore passes through X, Y, Z .

Hence it must pass through the remaining vertex O , i.e. the centre of the hyperboloid.

If we reciprocate this theorem w.r.t. O ,

- (1) the hyperboloid becomes a new hyperboloid, centre O , with its asymptotic cone possessing three mutually perpendicular generators;
- (2) $ABCD$ becomes a tetrahedron $abcd$ self-polar w.r.t. this new hyperboloid;
- (3) the sphere $ABCD$ becomes a paraboloid of revolution touching the faces of $abcd$ and having O as focus.

Now the foot of the perpendicular from the focus of a paraboloid of revolution upon any tangent plane lies on the tangent plane at the vertex.

Hence the feet of the perpendiculars from O to the faces of $abcd$ lie in a plane (which we may call a "Simson Plane").

Thus, if a tetrahedron $abcd$ is self-polar w.r.t. a hyperboloid whose asymptotic cone possesses three mutually perpendicular generators, then the centre of the hyperboloid has a Simson Plane w.r.t. the tetrahedron $abcd$.

The above pair of theorems provide an extension to three dimensions of the fact that if a triangle is self-polar w.r.t. a rectangular hyperbola, then its circumcircle passes through the centre of the rectangular hyperbola.

It may be noted that the locus of a point which has a Simson Plane w.r.t. a tetrahedron $ABCD$ is the cubic surface

$$\frac{1}{xh_1^2} + \frac{1}{yh_2^2} + \frac{1}{zh_3^2} + \frac{1}{wh_4^2} = 0,$$

where $(xyzw)$ are the tetrahedral coords. of the pt. w.r.t. $ABCD$ and $h_1 h_2 h_3 h_4$ are the perpendicular heights of the tetrahedron.

H. LOB.

667. [T. 4. a; V. 5.] Connection between a Cooling Curve and the Slide Rule.

Newton's Law of Cooling tells us that the rate of cooling of a heated body is proportional to its excess of temperature above that of its surroundings. Suppose that y represents the excess and that a graph is drawn coordinating y and the time t .

Then, by Newton's Law, $\frac{dy}{dt} = -ky$;

$$\therefore \log y = -at + b, \text{ where } a \text{ and } b \text{ are constants.}$$

Hence the difference of the logarithms of any two ordinates bears a constant ratio to the length cut off by them on the time axis.

So we see that if the feet of the ordinates on the time axis are labelled as the excesses of temperature represented by the ordinates respectively, the time axis becomes a Slide Rule.

A. W. LUCY.

668. [D. 1. b. a.] Note on the Fourier Series for a Linear Function.

Professor Sommerville's article on "An Analytical Remainder Formula" opens the way to a simple method of expressing a linear function as a Fourier Series. The method has the merit of showing the origin of the discontinuity in the series.

Consider the sequence of numbers

$$1, 2, 3, \dots (p-1), 0, 1, 2 \dots (p-1), 0 \dots$$

In Sommerville's notation the n th number of the sequence is $R(n/p)$. The differences between consecutive numbers of the sequence are

$$1, 1, 1 \dots 1, -(p-1), 1, 1, 1 \dots 1, -(p-1), 1, 1 \dots$$

Writing a for $2\pi/p$, we notice that the expression

$$-[\cos ra + \cos 2ra + \cos 3ra + \dots + \cos (p-1)ra]$$

is equal to 1 for all integral values of r except such as are multiples of p , and for such multiples of p the expression is equal to $-(p-1)$.

Accordingly, the sum of n such expressions is $R(n/p)$,

$$\text{i.e., } R(n/p) = - \sum_{r=1}^{p-1} \left[\sum_{q=1}^{n-1} \cos qra \right]. \quad (1)$$

As the limits are finite, the order of summation may be changed, and

$$R(n/p) = - \sum_{q=1}^{\lfloor n/p \rfloor - 1} \left[\sum_{r=1}^{p-1} \cos qra \right]. \quad (2)$$

$$\text{Since } \sum_i^n \cos qra = \frac{\sin(n + \frac{1}{2})qa - \sin(\frac{1}{2})qa}{2 \sin \frac{1}{2}qa} = \frac{1}{2} \left[\frac{\sin(n + \frac{1}{2})qa}{\sin \frac{1}{2}qa} - 1 \right], \quad (3)$$

$$\text{it follows that } R(n, p) = \frac{p-1}{2} - \frac{1}{2} \sum_{q=1}^{p-1} \frac{\sin(n + \frac{1}{2})qa}{\sin \frac{1}{2}qa}. \quad (4)$$

This is the answer to Prof. Sommerville's problem.

For our purpose we write $(n + \frac{1}{2})a = \theta$, and remembering that $pa = 2\pi$, we notice that $[R(n, p) + \frac{1}{2}]a = R(\theta, 2\pi)$, the remainder when θ is divided by 2π .

$$\text{Accordingly, } R(\theta, 2\pi) = \pi - \frac{\pi}{tp} \sum_{q=1}^{p-1} \frac{\sin q\theta}{\sin \frac{q\pi}{p}}, \quad (5)$$

provided that θ is an odd multiple of π/p .

To derive a more general formula, we quote the expansion

$$\begin{aligned} \frac{\pi}{p} / \sin \frac{q\pi}{p} &= \frac{1}{q} - \frac{1}{q+p} + \frac{1}{q+2p} - \dots \\ &\quad - \frac{1}{q-p} + \frac{1}{q-2p} - \dots, \end{aligned} \quad (6)$$

By substituting from (7) in (5) and rearranging the terms, we find that

$$R(\theta, 2\pi) = \pi - \sum_{q=-\infty}^{q=\infty} \frac{\sin q\theta}{q}, \dots \quad (8)$$

and therefore $R(\theta, 2\pi) = \pi - 2 \sum_{q=1}^{\infty} \frac{\sin q\theta}{q}$(9)

This relation has been proved under the condition that θ is commensurable with π . The last step, the removal of this restriction, depends on the fact that in general $\sum_0^{\infty} \frac{\sin q\theta}{q}$ is a uniformly convergent series, and therefore represents a continuous function.

When θ is an exact multiple of 2π , the equation (9) does not hold. The right-hand side of the equation becomes π , which is equal to the limit of

$$\frac{1}{2} \cdot [R(2m\pi + \epsilon, 2\pi) + R(2m\pi - \epsilon, 2\pi)].$$

The necessity for an appeal to the theory of uniform convergence makes the proof not so simple as it promised to be. I expect that in Prof. Hardy's scheme of giving marks to proofs of analytical theorems it would come out badly.

The alternative proof based directly on equation (5), in which equation p is made to approach infinity, should I think be marked more severely however.

I hope that if any reader of the *Gazette* is moved to draw accurate graphs of the function on the right of equation (5) he will publish his curves and discuss the laws governing their oscillations. F. J. W. W.

F. J. W. W.

669. [v.] *Mathematical Symbols.*

Walkingame (fl. 1750-1785), *The Tutor's Assistant*, 20th edition, 1784:

$\sqrt{}$ prefixed to any number signifies the Square Root of this number is required.

V³ signifies the Cube or 3rd Power.

V^4 denotes the Biquadrate or the 4th Power, etc.

Facit used for = . 7695 @ $\frac{1}{2}$ d. facit £16. 0s. 7 $\frac{1}{2}$ d. He also uses the sign =.
Ditto, First Preface 1827, 6th edition 1830, edited by William Birkin, Master
of an Academy in Derby:

σ (cap. S) for difference when uncertain which is the greater

Thomas Dilworth, *The Schoolmaster's Assistant*, 11th edition, 1762:

+ St. George's Cross signifies *more*, or Addition.

\div :: A Line between two Points, or between four Points, is the sign of Division.

✓ or \sqrt{q} Prefix to any Number supposes that the Square-Root of that Number is required. Sometimes it is the sign of Irrationality, and signifies that the Square-Root of such a Number can never be truly found.

/c. As above, but for "Square" read "Cube."

He uses *Facit* for = or for *Ans.*

Barrow (*Euclid*) uses $|-|$, the Difference or Excess; also that all the quantities which follow are to be subtracted, the signs not being changed.

[Apropos of the use of capital letters for economical reasons, e.g. > for >; it may be noted that authors in the eighteenth century complained of the meanness of the Cambridge University Press for using daggers set sideways instead of the usual +.] W. J. G.

REVIEWS.

Principles of Geometry. Vol. II. **Plane Geometry : Conics, Circles, Non-Euclidean Geometry.** By H. F. BAKER. Pp. 243. 15s. 1922. (Cambridge University Press.)

The first avowed aim of this book is the modest one : "to put the reader in touch with the main preliminary theorems of plane geometry"; and the second is "to test the application in detail of the logical principles explained in Volume I. . . . to bring to light the assumptions which underlie an extensive literature in which coordinates are freely used without attempt at justification." As to the first, one wonders what a reader, not already in touch with plane geometry, would make of it all. The arrangement is logical, not didactic, and the matter is so compressed as to be no easy reading on any terms. But it is the second aim that caused the book to be written, and this is most satisfactorily carried out. The present volume justifies the principles of the earlier part, by deducing from them the main properties of conics and circles.

After some Preliminary remarks, two series of "examples" of 40 pages each, in Chapters I. and III., include most of the contents of our conventional books on geometrical and analytical conics, on the unconventional basis of excluding the idea of congruence. This idea is dealt with in Chapters IV. and V., which contain a most refreshing discussion on coordinates and distance. Among other topics discussed are : plane geometry regarded as a projection of geometry on a quadric, the formulae of spherical trigonometry, and, in two notes, certain important configurations of straight lines.

The book should be read, not reviewed, and to this end the following quotations may stimulate :

"It is admitted that, by help of the symbols $x+iy$, we can define a closed system of points and figures to which this closed system of symbols is appropriate. But this does not appear to preclude the existence of other points and figures susceptible of geometrical theory"; though unfortunately this is precluded in the present volume owing to assumptions such as Pappus' theorem.

"Distance, as usually understood, . . . so far from being an obvious notion, characterising the space of experience, appears as an artificial, if useful, application of the algebraic symbols."

"No deduction of a really geometrical kind can be legitimately based on statements of which any particular conception of distance forms an indispensable part; such statements are equivalent only to statements in regard to the behaviour of particular measuring instruments, which must rest on physical hypotheses." This amounts to a definition of "really geometrical," according to which the only really geometrical property of a circle is that two of its points have been labelled *I* and *J*.

It is a great advantage to be made to think about common ideas which are usually taken for granted, especially when, as in the present case, the discussion can be followed without having to learn an entirely new language. We are grateful for the substantial addition this volume makes to our literature of geometry.

H. F. H.

Two-Figure Tables. Compiled by C. R. G. COSENS and printed on card $10'' \times 4\frac{3}{4}''$. 6d. each, 4s. per dozen. (Bowes and Bowes, Cambridge.)

Many boy who can trace accurately a curve with whose theoretical form he is acquainted, be it even a conic referred to arbitrary axes, makes a poor show at plotting from an unfamiliar formula, however straightforward, for practice in this kind of work tends to demand arithmetical labour out of all proportion to its value. These tables are designed to remove this difficulty; on the two sides of a single card with rounded corners are given the values, correct to within 1 per cent., of

$$\begin{aligned} x^2, & x^3, & x^{\frac{1}{2}}, & x^{\frac{1}{3}}, & x^{\frac{2}{3}}, & 1/x, & \log_{10}x, & \log x, \\ e^x, & e^{-x}, & \sin x, & \cos x, & \tan x, & \sinh x, & \cosh x, & \tanh x, \end{aligned}$$

for values of x at intervals of 0.1 from 0.0 to 5.0 and of 0.5 from 5.0 to 10.0.

The cards will be extremely useful not only in schools, but also in technical institutes and for evening classes, and the publishers are to be congratulated on carrying out admirably a most excellent idea. E. H. N.

The Mathematical Theory of Relativity. By A. S. EDDINGTON. Pp. ix + 247. 20s. net. 1923. (Cambridge University Press.)

The first draft of this important treatise was published in 1921 as a mathematical supplement to the French edition of the author's *Space, Time and Gravitation*, but the book has received so many additions that it is now three times the size of its 1921 form, which itself was really an enlarged version of the *Report on the Relativity Theory of Gravitation*. It is instructive to compare the earliest exposition with the latest, which is distinctly more cautious. It is candidly admitted that there is a certain hiatus in the argument. This hiatus is in the introduction of coordinates. With great labour Robb has filled it up as far as the restricted theory is concerned, but Prof. Eddington says that it is difficult to see how the problem can be attacked in the general theory. Of course Prof. Whitehead's recent work appeared too late to be taken into consideration.

The Principle of Equivalence, which asserts the equivalence (at a point) of a gravitational field of force and a field of force due to a transformation of coordinates, is now deposed from the high position it formerly held. It is useful in suggesting results to be tested by the tensor calculus, but some of the results turn out to be definitely wrong. At one time it was supposed that a rod would contract if placed radially in the gravitational field of a massive particle, but not if placed tangentially. This was deduced from Schwarzschild's interval formula. It is now pointed out that results which depend upon a particular choice of coordinates have no observational significance. The coordinates introduced by F. W. Hill and G. B. Jeffery are given the name of *isotropic coordinates*, and are shown to be those which, strictly speaking, should always be used for measurements made in a terrestrial laboratory. In this system the orientation has no effect upon the coordinate length of a small rigid rod or upon the velocity of a ray of light. Prof. Eddington refuses to make any prediction as to what would happen to a material rod set in motion, as the impulse required to start it might cause temperature effects.

However, it must not be supposed that Prof. Eddington's faith in relativity is weakening. He believes that the old conceptions have become untenable and that the new ideas throw a clear and penetrating light on the origin and significance of the great laws of physics, though quantum phenomena and the structure of the electron remain in darkness. He claims that the tensor calculus is not, as some may think, an evil necessity, to be replaced as soon as possible by simpler methods, but that it is of great importance as corresponding precisely to our experimental knowledge. This knowledge is usually expressed in terms of physical quantities which are *manufactured articles*, arising in part from more or less arbitrary conventions of the experimenter. The difficulties of the tensor calculus have been greatly reduced by setting out the work rather fully. This subject is now accessible to the average mathematician with sufficient time and energy. To master it requires doggedness rather than any special knowledge or ability.

Weyl's work is spoken of with deep respect, and is closely followed in several paragraphs. *Tensor-densities* are used for *quantities* (e.g. momentum) expressed as so much per unit mesh, as opposed to *tensors* for *intensities* (e.g. field of acceleration) expressed as so much at a point. It is shown that the conservation of energy is incompatible with the general theory of relativity, as what corresponds to potential energy is not a tensor or a tensor-density and has no physical reality. It can be made zero or non-zero at any point by a suitable choice of coordinates. Potential energy is declared to be a mere mathematical fiction, and the rejected law of conservation is replaced by a new law showing what becomes of kinetic energy when it is not conserved. This new law is said to be simpler than the old from the absolute point of view, but no examples of its application to problems are given. Perhaps the old law is sufficiently accurate to account for all observed facts. No doubt Weyl and other continental writers consider any failure in the conservation laws amply atoned for by the establishment of a *Principle of Stationary Action*,

by which the laws of the universe are obtained by equating to zero the variation of an integral. But on this point Prof. Eddington dissents strongly, and shows that the principle as generalised in relativity is untrue. "In fact action is only stationary when it does not exist—and not always then." A new conception, which is called *Hamiltonian differentiation*, seems to be more important.

As regards Weyl's non-Riemannian geometry, a curious change of position seems to have taken place. This theory was originally put forward as expressing the actual physical effect of an electromagnetic field upon a material rod, and it was explicitly stated that the absence of such a field was the condition necessary for Einstein's theory to be valid. This was challenged by Prof. Eddington, who regarded Weyl's work ("unquestionably the greatest advance in the relativity theory after Einstein's work") as a valuable method of visualising certain mathematical relations, but not as a statement of physical facts. It is stated that Weyl himself has now come round to this view. A further generalisation, due to Prof. Eddington himself, gives new insight into the meaning of the less general geometry which is sufficient for Einstein's theory.

No reviewer thinks that he has done his duty unless he has found something to criticise. In the present case it is difficult to find any fault, but it may be mentioned that the treatment of experimental work is rather scanty. Sommerfeld's relativity explanation of the *fine structure* of certain spectral lines is not dealt with at all. The subject of rotation, to which non-relativists attach great importance, receives very little attention. On p. 99 we find an interesting debate on rotation between a relativist and a non-relativist. The latter appears to be holding his own rather well, when the ground is suddenly cut from under his feet by the supposition that the facts are widely different from what they actually are. This is hardly giving the non-relativist devil his due.

As is usual with Prof. Eddington, the whole of the book is expressed in a most delightful fashion, illuminated by occasional flashes of wit. It has the advantage of the beautiful printing of the Cambridge University Press. Taken in conjunction with *Space, Time and Gravitation*, it forms the best exposition of relativity that has yet appeared.

The Theory of Spectra and Atomic Constitution. By NIELS BOHR. Pp. x + 126. 7s. 6d. net. 1922. (Cambridge University Press.)

This book is a translation of three addresses by Professor Bohr. These are non-mathematical and in some places rather indefinite, but they give an outline of the origin and development of his theories. The first address was delivered in 1913, soon after the publication of his first ideas concerning spectral lines. The second (1920) presents these ideas with important developments.* Much use is made of the important *Correspondence Principle*. This is difficult to state concisely, but we may say in general terms that it postulates a correspondence between the quantum theory and the older electromagnetic theory which has the effect of retaining the older theory and all its valuable results for long wave-lengths. Unless this principle is introduced, the quantum theory has no means of predicting the intensity and state of polarisation of the spectral lines, and, what is worse, predicts the existence of lines which appear to be absent.

The third address (1921) contains a wonderful application of quantum theory to explain the physical and chemical peculiarities of all the elements. The *Periodic Classification* has long held a prominent place in chemistry. Bohr presents it in a new and greatly enlarged form, and he tries to account for valency, colour, magnetic and electro-chemical properties by considering the number and distribution of the electrons revolving round the nucleus. The theory is still very incomplete, even as regards its fundamental principles, but it is astounding that it should be possible to include such a large number of apparently disconnected empirical results in a system of a comparatively simple kind. The detailed mathematical investigation of this system offers great scope for research.

H. T. H. PIAGGIO.

* An introductory account of Bohr's theory was given in the review of Reich's *Quantum Theory* (*Mathematical Gazette*, March, 1923).

Elementary Calculus. By FRED. S. WOODS and FRED. H. BAILEY. Pp. viii + 318. 13s. 6d. 1922. (Ginn and Co.)

This book is written for first year students at an American Institute of Technology. It contains no serious attempt to deal with the real difficulties of the subject. The treatment of limits is unsatisfactory. Considering the liberal assumptions that the Authors are prepared to make with regard to convergence, a simpler theory of the logarithmic and exponential functions might well have been devised. Hyperbolic functions are not introduced, so that integrations involving $\sqrt{x^2 \pm a^2}$ are in the inconvenient logarithmic forms. Too many of the integrals are merely stated for verification; in fact the indefinite integration is quite unsystematic. Some chapters containing applications in which reference to first principles is unnecessary are much more attractive, and there is a good collection of examples. A. ROBSON.

Education on the Dalton Plan. By H. PARKHURST. Pp. xvi + 224. 5s. net. 1922. (Bell & Sons.)

Dalton Plan Assignments. Vol. II. **Mathematics and Science.** Compiled by the Staff of Streatham County Secondary School for Girls. Pp. 70. 2s. net. 1922. (Bell & Sons.)

So much has been said and written about the Dalton Plan that there is no need to dwell upon the contents of these little volumes. To those who as yet are unfamiliar with the details of the Plan, and who wish to know how far those who are trying it have been successful, there can be no better introduction than the above-named books. The typical "grouser" will tell us that after all the Dalton Plan is just the plan of the "crammer" of old, naturally adopted by him when there was plenty of time for the process of cramming. What we shall very much like to have in the *Gazette* is a detailed report from such boys' secondary schools as have made the experiment over a period, say, of a couple of years. Considering the appalling waste of educational effort which confronts every teacher of experience, some such plan as the Dalton is bound to come.

Calculus and Probability for Actuarial Students. By A. HENRY. 12s. 6d. net. Pp. 152. 12s. 6d. net. 1922. (C. & E. Layton.)

First and foremost in the work of the student of actuarial science comes the subject of Finite Differences. The author of this manual tells us that, in spite of the theoretical objections, it is found both necessary and convenient "to treat the fundamental propositions of the Differential and Integral Calculus as being, substantially, special cases of similar propositions in Finite Differences." Thus we find about sixty pages given to the former subject and as many to the calculus, the remaining thirty pages or so being devoted to a simple exposition of the numerical or "frequency" theory of probability. The author has thus been compelled to "include only such problems as are requisite for a proper knowledge of the subjects within the syllabus." One does not feel quite happy in assisting at maimed rites, but given these inevitable limitations the author has done his work well. The text is written in clear and succinct style, and the illustrative examples are judiciously chosen. It remains to add that the publishers are producing the volume by the authority and on behalf of the *Institute of Actuaries*.

A New Manual of Logarithms to Seven Places of Decimals. Edited by DR. BRUHN. Thirteenth stereotyped edition. Pp. xxiv. 610. 12s. 6d. net. 1922. Printed in Germany. (Chapman & Hall.)

"BRUHN, 1870. Seven figure logarithms of numbers to 1000, and from 10,000 to 100,000, with differences and *all* the proportional parts. The *all* is printed in italics, because in Babbage, Callet, etc., only every other table of proportional parts near the beginning of the table is given, for want of space."

In this work there is no inconvenient crowding, as even where the side-tables are very numerous, the type, though small, is still very clear. The constants *S* and *T*, for the calculation of signs and tangents, are added, and placed at the bottom of the page, as also are the degrees, minutes and seconds in every tenth number of the number-column (regarded as that number of seconds), and the same for each of these numbers multiplied by ten.

Log sines, cosines, tangents and cotangents to every second from 0° to 6° to seven places, with differences throughout, and *proportional parts*, except in the portion of the table from 10° to $1^\circ 20'$, where the size of the page would not admit of their insertion.

Log sines, cosines, tangents and cotangents from 6° to 45° to every ten seconds, to every seven places, with differences and proportional parts. Of course room could not be found for the proportional parts of all the differences; but throughout all the table on no page are there less than six proportional-part tables.

On p. 186 the first hundred multiples of the modulus and its reciprocal are given to ten places; and at the end of the book are tables of circular arcs, viz. the circular measure of $1^\circ, 2^\circ, \dots, 180^\circ, 1', 2', \dots, 60', 1'', 2'', \dots, 60''$ to ten places, an age for the conversion of arc into time, and some constants. In T.I. the change in the line is denoted by a bar placed over the fourth figure of all the logarithms affected, the similar change when the third figure is decreasing being denoted in the other tables by an asterisk; a final 5 increased has a bar superscript. It is incorrectly stated in the preface that the practice of marking all the last figures that have been increased, was introduced by SCHRÖN; for this invention was due to Babbage (see his preface, p. x). Dr. Brumns may, however, merely mean that the mark (viz. a bar subscript) introduced by SCHRÖN (1860) fatigues the eye and is of next to no use; and if so, we entirely agree with him. In Babbage the increase is denoted by a point subscript, which the reader scarcely notices; but in Schrön the bar catches the eye at once and is confusing. The cases also in which it is necessary to know whether the last figure (unless a 5) has been increased are excessively rare; and, in fact, any one who wants such accuracy should use a ten-figure table.

On the whole, this is one of the most convenient and complete (considering the number of proportional-part tables), logarithmic tables for the general computer that we have met with; the figures have heads and tails; and the pages are light and clear. Further, we believe it is published at a low price.

The above was written by Mr. J. W. L. Glaisher for the British Association Report on Mathematical Tables, 1873.

As the work is stereotyped, we brave all risks of infringing copyright, and print without apology the *ipsissima dicta* of a famous veteran, the greatest living expert on the subject. All we need add is that the pages are about ten inches by six and a half.

La Composition de Mathématiques dans l'Examen d'Admission à l'Ecole Polytechnique de 1901-1921. By F. MICHEL and M. POTRON. Pp. xii + 452. 40 fr. 1922. (Gauthier-Villars.)

Some twenty years ago M. Michel published a similar volume containing the questions set at the Concours from 1860 to 1900. The arrangement is as follows. The first half of the volume contains, in chronological order, each question set in the Concours. The various steps in the solution of each are given, but not in detail. The second part, arranged according to subjects, contains, with all necessary cross references, solutions, or hints to solution, of all the minor problems that have arisen in the questions of the first part, and of which the results alone had there been provided. Thus the second part contains a large collection of exercises on applications of general methods, which it is eminently desirable that every student should be able to solve with rapidity and certainty. The ground covered comprises: Algebra, Trigonometry, Analysis, Analytic Geometry of two and three dimensions, Kinematics, and the Dynamics of a Point.

168. Numbers. When Boswell had purchased a farm, Johnson "made several calculations of the expense and profit; for he delighted in exercising his mind on the science of numbers." It may be remembered that on Tuesday, Aug. 31, 1773, Johnson gave the daughter of his landlord a book which he had bought at Inverness. Many questions were asked about this book by those who were inquisitive to know the kind of book Johnson carried about with him, and the announcement that it was *Cocker's Arithmetic* generally raised a laugh, which irritated the great Cham. Boozzy, at General Oglethorpe's, ventured to ask the great man why he bought such a book at Inverness. "He gave me a very sufficient answer. 'Why, sir, if you are to have but one book with you on a journey, let it be a book of science. When you have read through a book of entertainment, you know it, and it can do no more for you; but a book of science is inexhaustible.'" Johnson considered that all minds are equally capable of "attaining the science of numbers" (Croker, p. 480, 1853), and was himself accustomed to indulge in arithmetical calculations to "calm his mind."

PROBLEMS AND SOLUTIONS.

So many solutions of the Problem in *Note 644* have been sent in that the Editor is tempted to reopen the column which was a prominent feature in earlier days of the history of the *Gazette*. To discover if the interest in "Problem-grinding" is sufficiently general, we shall first reprint only the unsolved problems from the old list.

SOLUTIONS.

[**K¹, 1. c.**] *Solutions to Note 644, Math. Gazette, xi. p. 173.*

$$(i) FA/AC = \sin 30^\circ / \sin 130^\circ = 1/(2 \cos 40^\circ) = \sin 40^\circ / \sin 80^\circ \\ = \sin BEC / \sin ECB = CB/CE = FB/BE;$$

and

$$\hat{F}AC = \hat{F}BE;$$

$\therefore \triangle FAC, FBE$ are similar, and $\hat{B}EF = \hat{A}CF = 30^\circ$.

J. L. BURCHNALL, S. INMAN, E. P. LEWIS, R. NETTELL,
A. V. RICHARDSON, C. W. M. SHERRIFF.

Mr. R. F. Davis, who gives a similar solution, notes the relation

$$8 \sin 10^\circ \sin 50^\circ \sin 70^\circ = 1,$$

which is derived at once from "my pet formula"

$$\sin 3\theta = 4 \sin \theta \sin(60^\circ + \theta) \sin(60^\circ - \theta),$$

continually suggested in the study of "Morley's Theorem."

Mr. J. L. Burchnall obtains the relation from the substitution $y+y^{-1}=2x$ in $y^6+y^3+1=0$, whence $x=\cos 40^\circ$ is a root of $8x^3-6x+1=0$, and

$$3 - 4 \cos^2 40^\circ = \frac{1}{2} \sec 40^\circ.$$

(ii) The following is simple, though not so neat as the official solution.

Take G in BA so that $BG=BE$.

Produce EG to H making $GH=BC$.

Then $\hat{A}EH = 60^\circ$; $\hat{AGH} = 80^\circ$.

$$AG = AB - AE = EC;$$

$\therefore \triangle AGH \cong \triangle EBC$;

$$\therefore \hat{A}HG = \hat{E}BC = 60^\circ;$$

$\therefore \triangle AHE$ is equilateral.

$$\therefore EH = BG.$$

$$\therefore EG = GF.$$

$$\therefore \hat{F}EG = 50^\circ \text{ and } \hat{F}EB = 30^\circ.$$

The same problem appears in the Entrance Scholarship Examination, Peterhouse and Sidney, Jan. 1916:

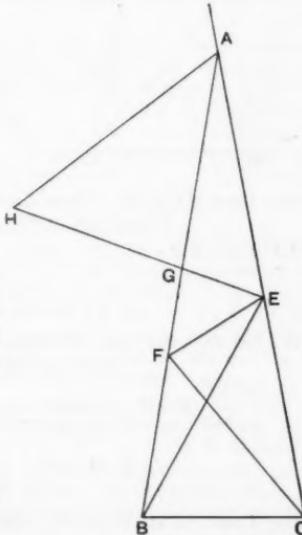
"On opposite sides of a base BC are described two triangles ABC, BCD , such that

$$\hat{A}BC = 30^\circ, \hat{A}CB = 80^\circ;$$

$$\hat{D}BC = \hat{D}CB = 50^\circ.$$

Show that, if AD is drawn, $\hat{D}AC = 30^\circ$, and find the other angles of the figure."

An interesting variation is to draw BE and CF making angles of 60° and 50° on the other side of BC to meet AC and AB produced. The drawing of the perpendicular as in the official solution of the original problem leads to a particularly neat solution.



In the course of a solution in *Mathematics from the Educational Times* (New Series, vol. xvii. p. 75) Mr. J. Blaikie takes a quadrilateral $ABCD$ with

$$\hat{A}BD = 30^\circ; \hat{DBC} = 20^\circ; \hat{ADB} = 40^\circ; \hat{BDC} = 60^\circ,$$

and proves that $\hat{DAC} = 30^\circ$; $\hat{ACD} = 50^\circ$, in both cases getting approximations by the use of logarithms. A slight rearrangement of the trigonometry would have shown that the values were exact.

A geometrical proof is easy, but I have had to use a property of the circle.

G. N. BATES.

(iii) *Solution by Elementary Geometry.*

Through B, C draw BD, CG parl. to CA, AB and meeting the parl. through F to BC in H, G . Produce BH, CF to meet in D .

Join GD . Let GH cut CA in K .

Then each of the figs. $BFGC, BHKC$ is a rhombus, and hence symmetrical w.r.t. either of its diagonals.

$$\hat{DBC} = 100^\circ; \hat{GBC} = 40^\circ;$$

$$\hat{HBF} = 20^\circ = \hat{FBE}. \dots \text{(i)}$$

$$\therefore \hat{HBG} = 60^\circ = \hat{CBE}.$$

$\therefore BE$ and BD are symmetrical w.r.t. BK .

$$\therefore BE = BG.$$

Again, DB, DG are symmetrical w.r.t. CF .

$$\therefore \hat{DGB} = \hat{DBG} = 60^\circ;$$

$\therefore \triangle BDG$ is equilateral.

$$\therefore BD = BG = BE, \dots \text{(ii)}$$

$$\hat{BDC} = 30^\circ. \dots \text{(iii)}$$

From (i) and (ii) D and E are symmetrical w.r.t. BF .

$$\therefore \hat{BEC} = \hat{BDF} = 30^\circ.$$

E. M. LANGLEY.

(iv) Draw $EL \parallel BC$. Then $\hat{LEB} = 60^\circ$, $AE = EB$, $BF = BC$.

$$LE/EB = LE/EA = BC/AC = 2 \cos 80^\circ;$$

$$LF/FB = (LB - BC)/BC = (EC - BC)/BC = (BC \sin 60^\circ / \sin 40^\circ - BC)/BC = 2 \cos 80^\circ \text{ (on reduction).}$$

$\therefore EF$ bisects \hat{LEB} , and $\hat{AEF} = 30^\circ$. W. MACKAY.

(v) Let BC = one unit of length. Then BF = one unit of length, and

$$BE = \sin 80^\circ / \sin 40^\circ = 2 \cos 40^\circ.$$

In $\triangle BEF$,

$$\tan \frac{F-E}{2} = \frac{2 \cos 40^\circ - 1}{2 \cos 40^\circ + 1} \cot 10^\circ = \tan 50^\circ \text{ (on reduction);}$$

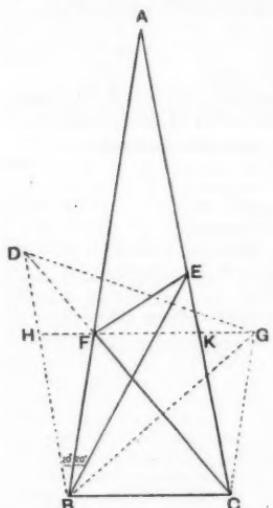
$$\therefore F-E = 100^\circ; F+E = 160^\circ. \therefore \hat{E} = 30^\circ.$$

J. W. MERCER and A. S. L. K (per S. LISTER).

(vi) Draw BG at 20° to BC , cutting CA in G . Then

$$\hat{GBF} = 60^\circ; \hat{BGC} = 80^\circ = \hat{BCG}. \therefore BC = BG.$$

Also $\hat{BCF} = \hat{BFC} = 50^\circ$; $\therefore BF = BG$ and $\triangle BFG$ is equilateral.



But $\hat{G}BE = 40^\circ = \hat{B}EG$. $\therefore BG = GE$. $\therefore GF = GE$.

And $\hat{F}GE = 40^\circ$. $\therefore \hat{G}EF = 70^\circ$ and $\hat{B}EF = 30^\circ$.

J. W. MERCER.

(vii) $B\hat{F}C = 50^\circ = \hat{B}CF$. $BF = BC$. Take C' in BE so that $BC' = BC$, and let I be the incentre of $\triangle BEC$. Then $\triangle BCC'$ is equilateral.

$$C'\hat{C}E = 80^\circ - 60^\circ = 20^\circ = \hat{C}EI, \text{ and } \hat{C}EC' = 40^\circ = \hat{ICE};$$

$$\therefore \triangle C'EC \equiv \triangle ICE.$$

$$IE = CC' = BC = BF; \therefore IE \parallel BF \text{ and } FE \parallel BI;$$

whence

$$F\hat{E}B = I\hat{B}E = 30^\circ.$$

R. NETTELL.

PROBLEMS.

171 [I. 2. b.], I. p. 89. If any number of consecutive integers from 2 onwards be expressed in prime factors, the sum of the indices of the factors will be more often odd than even.

275 [B. 4. c.], I. p. 217. Two light rods AB and AC freely jointed at A , rest in a vertical plane; B and C are in contact with a smooth horizontal plane. Two other light rods DE , EF are rigidly connected at E , and hang with a heavy body supported at E , their ends D and F carrying smooth rings sliding on AB and AC respectively. Required the stress at A , and the tensions in the rods DE and EF . (C.)

279 [J. 1. b.], I. p. 217. m points are taken on each of n straight lines, and all the points are joined by straight lines. Find the total number of intersections of lines in the figure. (C.)

283 [J. 1. b.], I. p. 233. A and B are two squads selected from n soldiers, so that (1) A is not to contain fewer soldiers than B , nor more than a ; (2) B is not to contain more than b soldiers, $b \leq a$. If the men are indistinguishable, find the number of arrangements. P. A. MACMAHON.

326 [I. 2. b.; J. 2. a.], I. p. 280. A product of two given numbers is tested by "casting out the nines," and the result fails to show that the product is wrong. What is the chance of its being right? J. F. HUDSON.

336 [L¹. 5. d.], I. p. 281. Find by polar coordinates the locus of intersection of two normals, the inclinations of which to the axis are $\phi, 2\phi$. A. S. TURNER.

337 [A. 1. c.], I. p. 281. If the greatest coefficient in the expansion of $(1+x)^n$ be a multiple of n and of $n-2$, then n is of the form $6p+1$ or $6p+3$. H. R. TYLER.

370 [M¹. 6. h. a.], I. p. 371. (i) A circle is described on a variable chord PQ through the cusp of a cardioid. Find the locus of the centres of similitude of this circle and the director circle of the cardioid.

[M¹. 5. c. β.], (ii) In a right cissoid, find a relation between the area of the generating circle and the areas of the first and second pedals with respect to a cusp. E. N. BARISIEN.

373 [I. 1.], I. p. 372. (i) Assuming that -1 has two square roots, prove (if possible) that it has no other square roots. (ii) Assuming that -1 has two and only two square roots, prove that the same is true of any number, real or complex. F. S. MACAULAY.

374 [L¹. 3. a.], I. p. 372. Find the equation to the axis of a parabola touching an ellipse at the extremities of a normal chord. E. M. RADFORD.

ERRATA.

- P. 191, l. 2 up and l. 8 up, for "isogonals," "isogonal," read "isoclinals,
"isoclinal."
- P. 231, l. 12, for "cannot be nines, eights, nor tens," read "can be nines,
but not eights or tens."
- " l. 29, for a read a_1 .
- " At foot of page add : "If the predecessors be merely 9's, equal digits
occur either when c_r is 0 and r is even, or when c_r is 9 and
 r is odd."
- P. 232, l. 5, delete comma after "to."
l. 8, for "1. 8. 7 = 3. 0. 8" read "1. 8. 7. 3. 0. 8."
- P. 263, l. 24, for "he is careful to prove" read "though he does not prove
this" (v. Heath's *Euclid*, Vol. II. p. 90).

THE LIBRARY.

THE Library has now been removed to 160 Castle Street, Reading, and Prof. E. H. Neville has taken over the duties of Honorary Librarian.

The Librarian will gladly receive and acknowledge in the *Gazette* any donation of ancient or modern works on mathematical subjects.

New Regulations for the use of the Library by Members will appear in the *Gazette* for July.

SCARCE BACK NUMBERS.

Reserves are kept of A.I.G.T. Reports and Gazettes, and, from time to time, orders come for sets of these. We are now unable to fulfil such orders for want of certain back numbers, which the Librarian will be glad to buy from any member who can spare them, or to exchange other back numbers for them :

- Gazette No. 8 (very important).
A.I.G.T. Report No. 11 (very important).
A.I.G.T. Reports, Nos. 10, 12.

UNIVERSITY OF LONDON.

A Course of Three Lectures on "PROBLEMS IN RELATIVITY" will be given by PROF. DR. H. A. LORENTZ, F.R.S. (Professor of Physics in the University of Haarlem), at UNIVERSITY COLLEGE, LONDON (Gower Street, W.C.) on Monday, June 4th; Tuesday, June 5th; and Thursday, June 7th, at 5.30 p.m. At the first Lecture the Chair will be taken by DR. A. N. WHITEHEAD, Sc.D., LL.D., F.R.S. (Professor of Applied Mathematics in the Imperial College of Science and Technology). *Admission Free, without Ticket.*

EDWIN DELLER, Academic Registrar.

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May, 1923.

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Constructive Arithmetical Exercises. Part I. with Answers. By R. W. M. GIBBS. Pp. xi + 244 + 44. 3s. 6d. net. 1922. (Blackie.)

Two Figure Tables. Powers, Roots, Logarithms, Transcendental Functions. Compiled by C. R. G. COSENS. Price 6d. 1922. (Bowes & Bowes, Cambridge.)

Cours de Physique Mathématique de la Faculté des Sciences. Compléments au tome iii. Conciliation. Un véritable Déterminisme mécanique avec l'existence de la Vie et de la Liberté morale. By J. BOUSSINESQ. Pp. xlvi + 247. 30 fr. 1922. (Gauthier-Villars.)

Common Sense of the Calculus. By G. W. BREWSTER. Pp. 62. 2s. net. 1923. (Clarendon Press.)

Constructive Mathematical Exercises. Based on A. E. LAYNG's Arithmetic. Part I. With Answers. By R. W. M. GIBBS. Pp. xi + 244 + 44. 3s. 6d. net. 1923. (Blackie.)

Atomes et Electrons. Rapports et Discussions du Conseil de Physique tenu à Bruxelles du 1^{er} au 6 Avril 1921. Pp. 271. 20 frs. 1922. (Gauthier-Villars.)

On the History of Caloric. By F. CAJORI. Pp. 483-492. Reprint from *Isis*, April, 1922.

Spanish and Portuguese Symbols for "Thousands." By F. CAJORI. Reprint from *The American Mathematical Monthly*. May, 1922.

Origin of the Names Arithmetical and Geometrical Progression and Proportion. By F. CAJORI. Reprint from *School Science and Mathematics*. xxii. 8.

The Mathematical Theory of Relativity. By A. S. EDDINGTON. Pp. ix + 247. 20s. net. 1923. (Cam. Univ. Press.)

An Introduction to the Principles of Mechanics. By J. F. S. ROSS. With an Introduction by Dr. WILLIAM GARNETT. Pp. x + 400. 12s. 6d. net. 1922. (Jonathan Cape.)

The Nebular Hypothesis and Modern Cosmogony, being the Halley Lecture, May 23, 1922, by J. H. JEANS. Pp. 31. 2s. 6d. net. 1923. (Clarendon Press.)

Plane and Spherical Trigonometry with Stereographical Projections. By J. A. BULLARD and A. KIERNAN. Pp. iv + 230. 1922. 6s. net. (Heath & Co.)

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Higher Geometry. By F. S. WOODS. Pp. x + 423. 22s. 6d. net. 1922. (Ginn & Co.)

The Dissection of Rectilinear Figures. By W. H. MACAULAY. Pp. 53-56. Reprint from *Messenger of Mathematics*. Aug. 1922.

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A Primary Classification of Projective Transformations in Function Space. Pp. 87-101. L. L. DINES. *A General Theory of Limits.* Pp. 102-121. E. H. MOORE and H. L. SMITH. *Substitution Groups whose Cycles of the same Order contain a given number of Letters.* Pp. 122-128. G. A. MILLER. *Boundary Value and Expansion Problems: Oscillation, Comparison and Expansion Problems.* Pp. 129-152. R. D. CARMICHAEL. *On a Theorem in General Analysis and the Interrelations of Eight Fundamental Properties of Classes of Functions.* Pp. 153-162. E. W. CHITTENDEN.

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Plane Involutions of Order Four. Pp. 163-171. T. R. HOLLCROFT. *Differential Equations with a Continuous Infinitude of Variables.* Pp. 172-190. I. A. BARNETT. *The Factorization of the Rational Primes in a Cubic Domain.* Pp. 191-203. *On the Structure of Finite Continuous Groups with a Single Infinitesimal Transformation.* Pp. 204-216. S. D. ZELDIN. *The Laplace-Poisson-Mixed Equation.* Pp. 217-224. K. P. WILLIAMS. *The Four Color Problem.* Pp. 225-236. F. FRANKLIN.

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May 1922.

Spanish and Portuguese Symbols for "Thousands." Pp. 201-202. *A General Construction for Circular Cubics.* Pp. 202-204. R. M. MATHEWS. *A Generalisation of the Strophoid.* Pp. 204-207. J. H. WEAVER. *Among my Autographs.* (Sir W. R. Hamilton.) Pp. 209-210. D. E. SMITH. *Solution of a Problem in Skeleton Division.* Pp. 211-213. D. R. CURTISS and A. A. BENNETT. *What is a Calculus?* Pp. 213-217. J. P. BALLANTINE and EDITOR.

Aug. 1922.

A Simple Form of Duhamel's Theorem and Some New Applications. Pp. 239-250. E. T. ETTINGER. *Note on Application of Diophantine Analysis to Geometry.* Pp. 250-252. H. L. OLSEN. *A Note on the Problem of the Eight Queens.* Pp. 252-253. W. H. BUSSEY. *Among my Autographs [Montucla].* Pp. 253-255. D. E. SMITH. *Discussions: A General Type of Reduction Formula.* Pp. 257-261. A. S. HATHAWAY and EDITOR.

Sept. 1922.

Infinite and Imaginary Elements in Algebra and Geometry. Pp. 290-297. R. M. WINGER. *Among my Autographs [Burckhardt, on Modern Teaching, and on Publication of Scientific Work].* Pp. 297-300. *Definitions of Imaginary and Complex Numbers.* Pp. 301-303. E. S. ALLEN.

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Contradictions in the Literature of Group Theory. Pp. 319-328. G. A. MILLER. "Statistics" in a Mathematical Encyclopedic Dictionary. Pp. 333-337. H. L. RIETZ. G. B. HALSTED. Pp. 338-340. F. CAJORI. *Among my Autographs [De Molire].* Pp. 340-343. D. E. SMITH. *Discussions: Graphical Solutions of Numerical Equations.* Pp. 344-346. W. H. BIXBY. *Note on the Irrationality of Certain Trigonometric Functions.* P. 346. R. S. UNDERWOOD. *Concyclic Points on an Equilateral Hyperbola and on its Inverses.* Pp. 347-348. R. M. MATHEWS.

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I. i. Oct. 1922.

Le reti di Tchebycheff sulle superficie ed il parallelismo nel senso di Levi-Civita. Pp. 1-6. L. BIANCHI. *Sulle funzioni analitiche d'ordine n.* Pp. 8-12. P. BURGATTI.

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Dimostrazione di un teorema relativo alla quartica gobba di 2¹ specie. Pp. 1-2. E. BERTINI. *Sui campi di estremali uscenti da un punto e riempienti tutto lo spazio.* Pp. 2-6. G. CARATHÉODORY. *Lo smorzamento delle trepidazioni nei meandri lubrificati di certi giunti e innesti meccanici.* Pp. 12-13. *Sui poligoni del Cremona.* Pp. 13-18. G. SUPINO.

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- Condition that a Tensor be the Curl of a Vector.* Pp. 425-427. L. P. EISENHART. *A New Generalisation of Tchebycheff's Statistical Inequality.* Pp. 427-432. B. H. CAMP. *Two Theorems on Multiple Integrals.* Pp. 433-435. P. FRANKLIN. *Kirkman Parades.* Pp. 435-437. F. N. CONANT. *Impossibility of restoring Unique Factorization in a Hyper-Complex Arithmetic.* Pp. 438-442. L. E. DICKSON. *A Revision of the Bernoullian and Eulerian Functions.* Pp. 443-450. E. T. BELL. *On the Existence of Curves with Assigned Singularities.* Pp. 451-455. J. L. COOLIDGE. *A Generalisation of Normal Congruences of Circles.* Pp. 456-462. J. L. WALSH.

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- On Circular Vertex Rings of Finite Section in Incompressible Fluids.* Pp. 112-140. N. SEN. *The Steering of an Aeroplane in a Horizontal Circle.* Pp. 141-150. N. BASU. *The Radius of a Circle in Homogeneous Coordinates.* Pp. 151-166. M. GHOSH. *Surface Waves due to a Submerged Elliptic Cylinder.* Pp. 167-174. S. MITRA. *An Algebraical Identity.* Pp. 175-176. P. O. UPADHAYA.

L'Enseignement Mathématique. (Gauthier-Villars.)

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- Un chapitre de méthodologie mathématique. Les imaginaires de Galois.* Pp. 249-268. M. STUYVAERT. *Sur des radicaux carrés.* Pp. 269-277. B. NIEWENGLOWSKI. *Démonstration du théorème de Liouville par l'élimination du temps entre les équations de Lagrange.* Pp. 277-285. E. TURRIERE.

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- Sur la cubique de M. C. Mateescu et le théorème de Simson.* Pp. 161-162. AD. MINEUR. *Despre Formula Crusterii or Finite.* Pp. 162-166. G. BRATU. *Comparatiune intre Produsul Numerelor Prime cu un Numar dat și inferioare lui și Produsul Divizorilor acelui Numar.* Pp. 166-167. I. LINTES.

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